

# The short-time behaviour of a kinetic Ashkin-Teller model on the critical line

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## Abstract

We simulate the kinetic Ashkin-Teller model with both ordered and disordered initial states, evolving in contact with a heat-bath at the critical temperature. The power law scaling behaviour for the magnetic order and electric order are observed in the early time stage. The values of the critical exponent  $\theta$  vary along the critical line. Another dynamical exponent  $z$  is also obtained in the process.

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It has been predicted by analytical calculations [1] and supported in Monte Carlo simulations [2, 3, 4, 5, 6] that there exists scaling in the macroscopic short-time regime for some critical dynamic processes. The universality was confirmed for the Ising model and the Potts model with various dynamics and different lattice structures [5, 7, 8]. Basing upon the universal scaling hypothesis for the initial stage, promising methods have been proposed for numerical measurements of critical exponents, including both static and dynamic ones [9, 10, 11]. It has been suggested that the critical point can be also determined in this stage [11]. The universal behaviour of the short-time dynamics is found to be quite general [12, 13, 14, 15], e.g., in connection with ordering dynamics or damage spreading [16, 17] and surface critical phenomena [18].

It is interesting to see the short-time behaviour of models possessing continuously varying critical exponents in equilibrium. It was found by Kadanoff and Wegner that there is a connection between the continuous variation of the critical exponents and the existence of a marginal operator [19]. The variation of critical exponents in the early time evolution implies that the operator keeps in marginal even in the non-equilibrium initial stage.

The Ashkin-Teller model is one of such models that has weak universality, i.e., the critical exponents vary with parameters of the interaction [20]. It contains two species of spins  $\{\sigma_i = \pm 1\}$  and  $\{\tau_i = \pm 1\}$ , located on a square lattice. The interaction is described by the Hamiltonian

$$H = \sum_{\langle i,j \rangle} [K(\sigma_i \sigma_j + \tau_i \tau_j) + K_4 \sigma_i \sigma_j \tau_i \tau_j] \quad (1)$$

This model is dual to the solved Baxter's eight vertex model. It has five phases. We only investigate the exactly known segment of the critical line that separates the disordered phase and the phase where both the magnetic order and electric order are not zero. Along that line, the critical exponents vary continuously. The critical line is given by the equation

$$\exp(-2K_4) = \sinh(2K) \quad (2)$$

where  $(\ln 3)/4 < K < +\infty$ . Besides the critical exponent  $\nu$  that governs the critical behaviour of the correlation length, there are other two independent critical exponents  $\beta_m$  and  $\beta_e$  corresponding to two order parameters, the magnetic order  $\langle \sigma_i \rangle$ , and the electric order  $\langle \sigma_i \tau_i \rangle$ , respectively. The critical exponents for the equilibrium state have been known as [20, 21, 22, 23]

$$\nu = \frac{2-y}{3-2y}, \quad \beta_m = \frac{2-y}{24-16y}, \quad \beta_e = \frac{1}{12-8y} \quad (3)$$

where the parameter  $y$ , ranging from 0 to  $4/3$ , is related to  $K_4$  by the equation

$$\cos(\pi y/2) = \frac{1}{2}[\exp(4K_4) - 1] \quad (4)$$

When dynamics is introduced into the Ashkin-Teller model, besides the well-known  $z$  that describes the divergence behaviour of the time-correlation length, there are two critical exponents related to the dimensions of two initial orders. As argued by Janssen et al., [1] the initial scaling emerges at fixed point under the renormalizational group transformation. By naive coarsing transformation, one can see two fixed points for initial states, i.e.,  $m_0 = e_0 = 0$  at very high temperature and  $m_0 = e_0 = 1$ . Our simulations are around these two fixed points.

The magnetic moment  $M^{(k)}$  and the electric moment  $E^{(k)}$  are defined as

$$M^{(k)} = \langle [(\sum_i \sigma_i)/L^2]^k \rangle_t, \quad E^{(k)} = \langle [(\sum_i \sigma_i \tau_i)/L^2]^k \rangle_t \quad (5)$$

In analogy to the Ising model, we suppose the following finite-size scaling relations hold in the vicinity of the first fixed point,

$$M^{(k)}(t, L, m_0) = b^{-k\beta_m/\nu} M^{(k)}(b^{-z_m} t, b^{-1} L, b^{x_m} m_0) \quad (6)$$

for  $e_0 = 0$ , and  $m_0$  small, and

$$E^{(k)}(t, L, e_0) = b^{-k\beta_e/\nu} E^{(k)}(b^{-z_e} t, b^{-1} L, b^{x_e} e_0) \quad (7)$$

for  $m_0 = 0$ , and  $e_0$  small. We have denoted the anomal dimensions of the initial magnetization and electric order as  $x_m$  and  $x_e$  respectively. For  $k = 1$ , taking  $b = t^{1/z}$  and assuming the initial orders are small enough, we can expand the magnetization(electric order) with respect to  $m_0(e_0)$  and obtain

$$M(t) = m_0 t^{\theta_m} F_m(t^{-1/z_m} L) \quad (8)$$

$$E(t) = e_0 t^{\theta_e} F_e(t^{-1/z_e} L) \quad (9)$$

where  $\theta_m = (x_m - \beta_m/\nu)/z_m$ , and  $\theta_e = (x_e - \beta_e/\nu)/z_e$ . The scaling functions  $F_m$  and  $F_e$  take account of the deviation from power laws due to the finite-size effect. They tend to constants as  $L$  tends to infinity and time tends to zero, but bigger than the microscopic time scale  $t_{mic}$ . It has been shown in various models that  $t_{mic}$  is ignorable in the heat-bath dynamic process. In the following fits, we will consider  $F_m$  and  $F_e$  as constants.

It has been stressed in Ref. [1] that the initial states must have very short correlation lengths and that a sharp preparation of the initial state improves the result. For the Ashkin-Teller model, there are two initial order parameters,  $m_0$  and  $e_0$ . So one has more than one way to approach the disordered fixed point. For instance, we can either let  $e_0 = 0$  and  $m_0$  small or reverse,  $m_0 = 0$  and  $e_0$  small, later on referred to as initial condition I and initial condition II, respectively. We find that the relaxation patterns from these two initial conditions are very

	$L = 180$	$L = 60$				$L = 30$
$y$	$m_0 = 0.02$	$m_0 = 0.02$	$m_0 = 0.04$	$m_0 = 0.06$	$m_0 = 0.08$	$m_0 = 0.04$
0	-.020(1)	-.021(4)	-.022(5)	-.019(5)	-.021(2)	-.022(6)
1/6	-.013(1)	-.014(2)	-.014(5)	-.013(4)	-.014(2)	-.015(6)
1/3	.009(2)	.008(1)	.007(5)	.007(3)	.007(3)	.007(4)
1/2	.041(5)	.041(3)	.041(5)	.041(2)	.041(2)	.039(6)
2/3	.085(4)	.085(4)	.085(5)	.084(3)	.083(2)	.086(6)
5/6	.137(2)	.137(2)	.134(3)	.136(3)	.134(1)	.135(3)
1	.191(2)	.189(2)	.188(3)	.188(2)	.185(1)	.187(3)
7/6	.230(1)	.229(2)	.227(1)	.225(1)	.223(1)	.225(3)

Table 1: Results for  $\theta_m$  with initial condition I of  $m_0 = 0.02$  on  $L = 180$ , and of  $m_0 = 0.02, 0.04, 0.06$ , and  $0.08$  on  $L = 60$ ,  $m_0 = 0.04$  on  $L = 30$ . In each case, the magnetization in the time interval  $1 \leq t \leq 5$  is used for the least-square fit to the power law.

different. However, for both initial conditions, power laws are observed. The exponents  $\theta$  depend on the initial conditions and vary with  $y$ .

The short-time behaviour of the magnetization with the initial condition I can be seen in Fig. 1. The points are magnetizations averaged over 40000 independent samples with  $L = 180$  and  $m_0 = 0.02$ . Since the two species of spins are symmetry, the effective sample number is doubled. Samples of each point are grouped into 4 runs, and the errors are obtained from them. The parameter  $y$  is increasing from the bottom to the top. The lines are curves of the power law Eq.(8) with the best-fit exponents  $\theta_m$  to the magnetizations in the first five time steps as given in Tab. 1. We stop at  $t = 5$  for that gives the smallest fluctuation. Tab. 1 also contains  $\theta_m$  for  $L = 30$ , and 60, which shows that the finite-size effects are smaller than the statistical fluctuation. By comparing results for various  $m_0$  from 0.02 to 0.08, on lattice  $L = 60$ , one can see that the  $\theta_m$  are quite stable. At the decoupled point  $y = 1$ , the Ashkin-Teller model reduces into two independent Ising models. Our best value  $\theta_m = 0.191(2)$  should be compared with the existing numerical results for the Ising model from Refs. [7, 17],  $\theta = 0.191(1), 0.191(3)$ , and from those obtained from auto-correlation before [2, 3].

Figure 2 shows the initial relaxation of the electric order starting from the initial condition II with  $e_0 = 0.02$ ,  $L = 180$  in the double-log scale. Each point is an average over not less than 80000 independent samples. To obtain stable results for  $y = 1$ , we used up to 480000 independent samples. The lines from the top to the bottom are curves of the power law Eq.(9) with the best-fit  $\theta_e$ , as given in Tab. 2, corresponding to  $y = 0, 1/6, \dots, 1$ , respectively. For  $y \leq 1$ ,

	$L = 180$	$L = 60$	$L = 30$
$y$	$e_0 = 0.02$	$e_0 = 0.02$	$e_0 = 0.04$
0	-.021(6)	-.022(5)	-.021(3)
1/6	-.036(6)	-.037(6)	-.036(3)
1/3	-.082(7)	-.085(5)	-.083(3)
1/2	-.169(6)	-.172(6)	-.171(3)
2/3	-.308(9)	-.312(11)	-.306(4)
5/6	-.510(9)	-.504(10)	-.498(5)
1	-.812(30)	-.793(29)	-.760(13)

Table 2: Results for  $\theta_e$  with initial condition II of  $e_0 = 0.02$  on  $L = 60$  and  $180$ , and of  $e_0 = 0.04$  on  $L = 30$ . The electric order in the time interval  $1 \leq t \leq 5$  is used for the least-square fit to the power law.

the order monotonously decreases with time in a power law. Discontinuity is found around  $y = 1$ , which is the turning point of coupling  $K_4$  from positive to negative. For  $y = 7/6$ , the order jumps to a negative value at the first time step. The fluctuation for the electric order is much bigger than that for the magnetization since the electric order decays rapidly to very small values. That is the reason why we have to measure the exponent  $\theta_e$  at the very beginning of the relaxation ( $1 \leq t \leq 5$ ).

Now, we turn to the measurement of the dynamic exponent  $z$ . Traditionally,  $z$  is measured from the long-time exponential decay of the time correlation or the magnetization of the systems [24, 25]. Due to the critical slowing down, this is somehow difficult. Recently, a few methods have been proposed to estimate  $z$  in the short time stage of dynamic processes. Stauffer suggested to obtain  $z$  from the power law decay of the magnetization with the known static exponent  $\beta/\nu$  as input. For a large enough lattice, one may expect a power law decay of the orders

$$M(t) \sim t^{-\beta_m/\nu z_m} \quad (10)$$

$$E(t) \sim t^{-\beta_e/\nu z_e} \quad (11)$$

before the exponential decay starts. For the Ising model and the Potts model with ordered initial states, it has been confirmed that the power law decay happens in the short-time regime [26, 27, 10, 11]. The exponent  $z$  may also be measured independently from the time-dependent Binder cumulants in the short-time regime with either ordered or disordered initial states as proposed in Refs. [9, 10]. Since higher order moments are needed, the results are relatively more fluctuating. With the initial dynamic exponent  $\theta$  in hand, Ref. [6] suggested to determine  $z$  from the power law behaviour of the autocorrelation. In this paper we do not

use this method since the  $\theta_e$  for the Ashkin-Teller model is big and difficult to be precisely evaluated.

	$L = 64$		$L = 90$	
$y$	$z_m$	$z_e$	$z_m$	$z_e$
0	2.23(4)	2.24(4)	2.21(2)	2.21(3)
1/6	2.27(6)	2.37(6)	2.24(5)	2.34(5)
1/3	2.31(2)	2.37(2)	2.29(4)	2.35(5)
1/2	2.31(4)	2.32(4)	2.26(3)	2.27(3)
2/3	2.26(2)	2.24(2)	2.24(4)	2.23(3)
5/6	2.19(2)	2.17(2)	2.20(1)	2.19(1)
1	2.17(1)	2.17(1)	2.15(2)	2.15(2)
7/6	2.12(3)	2.14(3)	2.14(3)	2.16(4)

Table 3: Results for  $z$  obtained by fitting the power-law decay of the order parameters from an ordered initial state with  $L = 64$  and  $L = 90$ , where  $z_m$  and  $z_e$  are corresponding to the magnetization and the electric order respectively. The exact values of  $\beta_m/\nu$  and  $\beta_e/\nu$  are used as input. To avoid the possible deviation from the power-laws in the beginning of the relaxation, the first 100 time steps are skipped in the fitting.

In Figs. 3 and 4, we plot the time evolution with the ordered initial state for the magnetization and the electric order respectively, in double-logarithmic scale. Each point is measured from 16000 samples on a lattice with  $L = 90$ . The curves are the power laws as given by Eqs. (10) and (11) with the best fit powers for various  $y$ . To avoid effects of  $t_{mic}$ , the first 100 time steps have been skipped. With the static exponents  $\beta_m/\nu$  and  $\beta_e/\nu$  given by Eq.(3), we obtain the dynamic exponent  $z_m$  and  $z_e$  from the powers of Fig. 3 and 4 respectively. The results are compared with those of  $L = 64$  in Tab. 3. The finite-size effects are about 2.5%, comparable with the statistical errors. As  $y$  varies from zero to 7/6, a remarkable change of  $z$  may not be accounted only to finite-size effects and statistical errors. This may indicate that the exponent  $z$  is varying. However, within the statistical errors, we can not distinguish between  $z_e$  and  $z_m$ . At the decoupling point  $y = 1$ , we obtain the dynamic exponent  $z = 2.15(2)$  from both magnetic and electric powers, which is consistent with recent result for the Ising model obtained by other authors [24, 25, 26, 27, 28, 29].

We also estimate the dynamic exponent  $z$  independent of the static exponents by the finite-size scaling fit of the time-dependent cumulants on lattices with  $L = 64$  and  $L = 90$ . The method has been described in detail in Ref. [10]. We get the mean value  $z = 2.16(6)$  from the global fit and  $z = 2.16(8)$  from the local fit. The large fluctuation prevents us to make any conclusion on whether  $z$  is

varying. Using the mean value  $z = 2.16$  as input, the static exponents  $\beta_m/\nu$  and  $\beta_e/\nu$  can be estimated from the finite-size scaling of the magnetization and the electric order respectively. The results turn out to be consistent with the analytic results with deviation smaller than 10%.

In conclusion, we have confirmed the initial scaling for the kinetic Ashkin-Teller model which is known to possess weak universality in the equilibrium. The power laws fit to the measurements remarkably well in the short-time evolution of the order parameters with both disordered and ordered initial states. The exponents  $\theta_m$  and  $\theta_e$  are found varying with  $y$ . The initial order *increase* is only observed for  $y > 1/6$  with the initial condition I. This fact can not be explained by the straightforward mean-field argument that claims the initial order increase should happen when the critical temperature is lower than the mean-field critical temperature. It seems that more careful analysis is needed.

For bigger  $y(> 1/6)$ ,  $e_0$  has negative dimension  $x_e$ . This means that the time scale associated with  $e_0$  is a short-time scale, in contrast with the long-time scale associated with  $m_0$ . In Fig. 2 one can see that the scaling window gets narrower and narrower as  $y$  increases (the dimension  $x_e$  decreases). When the time scale of  $e_0$  is comparable with the microscopic time scale, one will not see the initial power law behaviour of the electric order. On the other hand, for  $y = 0$  the electric order and the magnetization have the same value for  $\theta$  within the fluctuations. This should be clear since at that point  $K = K_4$ , i.e., there is symmetry between two orders.

The data in Tab. 3 indicate that the exponent  $z_m$  is approximately equal to  $z_e$ . This may imply that the model has only one time correlation length which goes to diverge at the critical temperature, as there is only one space correlation length in the equilibrium Ashkin-Teller model as indicated by the equality of  $\nu$  for two orders.

The exponent  $z$  may vary with  $y$ , however, the errors are still too big to allow a definite conclusion. There are still some interesting questions left for further study. For instances, one should better understand the influence of the choice of the initial conditions, the discontinuity of the electric order at the decoupling point, the tricritical point  $y = 0$ , etc. To locate the critical line segments which are not exactly known would be also a good way to test the power of short-time dynamics.

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## Figure captions

1. Magnetization versus time for the initial condition I with  $m_0 = 0.02$ ,  $e_0 = 0$ , and  $L = 180$ . The curves from the bottom to the top are best fits to the power law Eq. (8) with  $y = 0, 1/6, 1/3, 1/2, 2/3, 5/6, 1$ , and  $7/6$ , respectively.
2. Electric order versus time for the initial condition II with  $m_0 = 0$ ,  $e_0 = 0.02$ , and  $L = 180$ . The curves from the top to the bottom are best fits to the power law Eq. (9) with  $y = 0, 1/6, 1/3, 1/2, 2/3, 5/6$ , and  $1$ , respectively.
3. The magnetization for various  $y$  on  $L = 90$ , left for relaxation from the ordered state. The curves are best fits to the power law Eq. (10) with  $z_m$  given in Tab. 3 and  $\beta_m/\nu$  given by Eq. (3) as input.
4. The electric order for various  $y$  on  $L = 90$ , left for relaxation from the ordered state. The curves are best fits to the power law Eq. (11) with  $z_e$  given in Tab. 3 and  $\beta_e/\nu$  given by Eq. (3) as input.

Figure 1

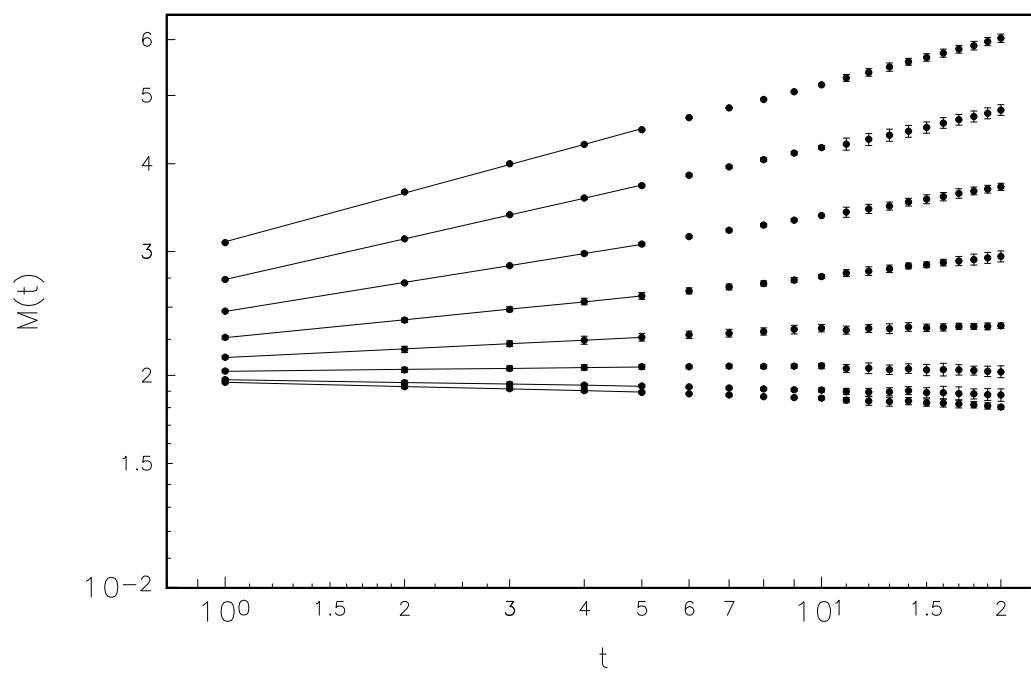


Figure 2

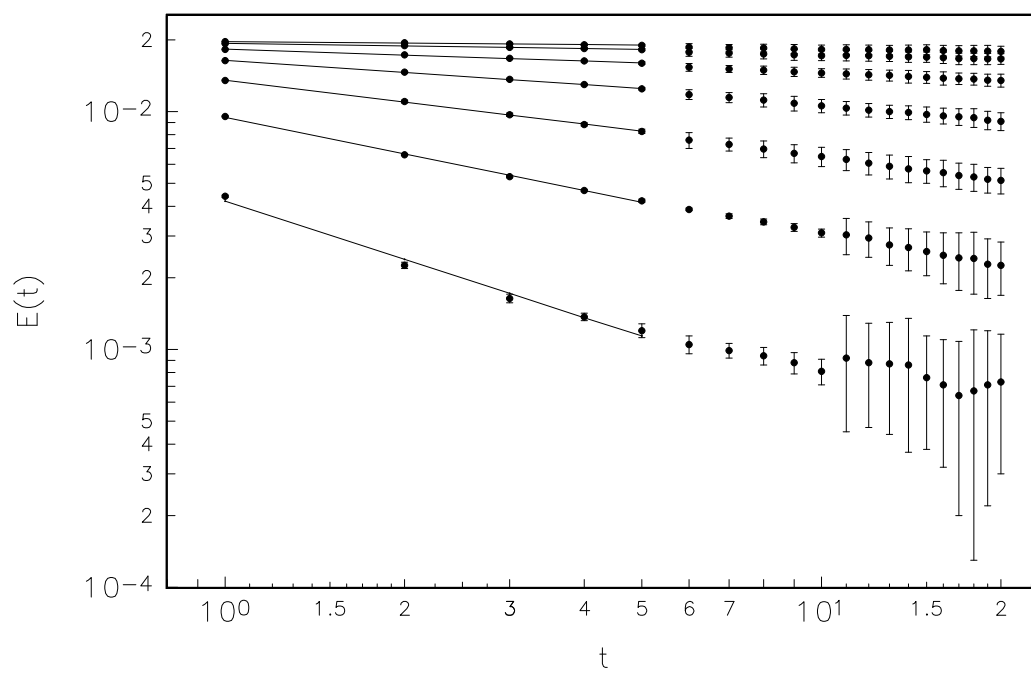


Figure 3

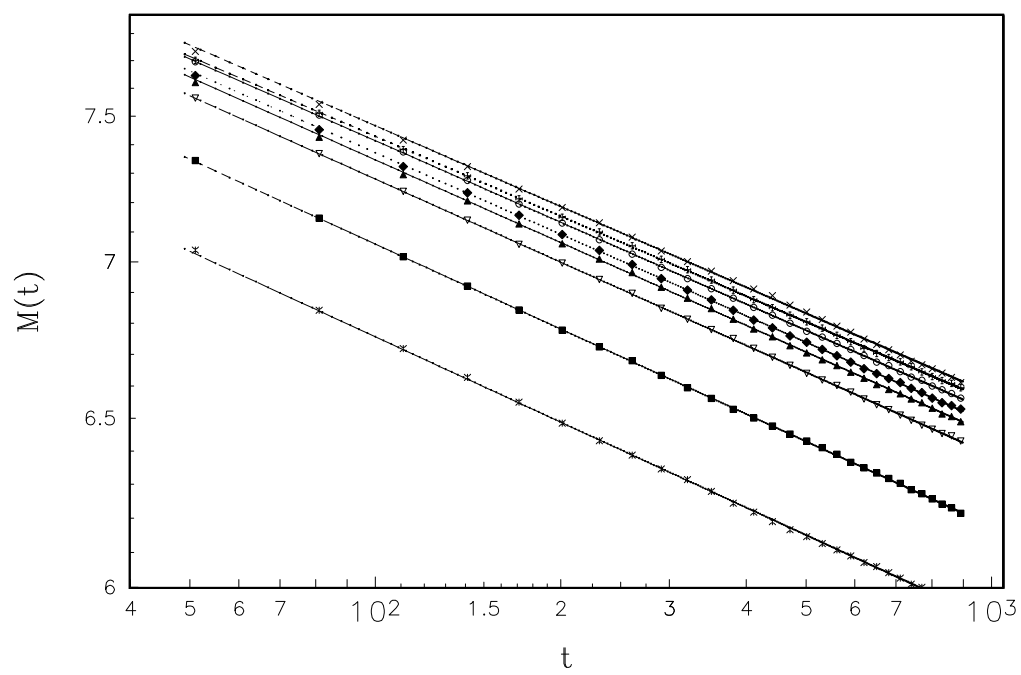


Figure 4

